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An Alternative Derivation of the Black-Sholes Model for Option Pricing

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Abstract: Investment enables investors to acquire shares in capital markets or their derivative assets, with options being one of the most common instruments. An option is a contract granting the right to buy or sell an underlying asset under specific conditions. The Black–Scholes equation is widely used for option pricing, though its derivation through Backward Stochastic Differential Equations (BSDEs) is less common. BSDEs are particularly relevant in incomplete markets, where not all options can be perfectly replicated. BSDEs also avoid the need to transform probability measures into the risk-neutral, thereby simplifying the pricing procedure. This study derives the Black–Scholes equation via BSDEs, modelling investor wealth and applying the Feynman–Kac theorem. The solution for call options is obtained by transforming the variables to get a diffusion equation, while put option prices are derived using the put–call parity principle. Finally, simulations of the Black–Scholes formula are conducted to analyse option price behaviour under varying volatility and risk-free interest rates.

Keywords: *Black-Scholes Equations, Option product, BSDE, Feynman-Kac theorem, Sensitivity analysis*

Abstrak: Investasi memungkinkan investor untuk memperoleh saham di pasar modal atau aset derivatifnya, dengan opsi menjadi salah satu instrumen yang paling umum. Opsi adalah kontrak yang memberikan hak untuk membeli atau menjual aset acuan dalam kondisi tertentu. Persamaan Black–Scholes banyak digunakan untuk penetapan harga opsi, meskipun derivasinya melalui Backward Stochastic Differential Equations (BSDEs) jarang diketahui. BSDEs sangat relevan di pasar yang tidak lengkap, di mana tidak semua opsi dapat direplikasi dengan sempurna, dan BSDEs juga menghindari kebutuhan untuk mengubah ukuran probabilitas menjadi risk neutral, sehingga menyederhanakan prosedur penetapan harga. Studi ini menurunkan persamaan Black–Scholes melalui BSDEs, dengan memodelkan kekayaan investor, dan menerapkan analogi teorema Feynman–Kac. Solusi untuk opsi beli diperoleh melalui transformasi variabel menjadi persamaan difusi, sementara harga opsi jual diturunkan menggunakan prinsip put-call parity. Simulasi rumus Black–Scholes dilakukan untuk menganalisis perilaku harga opsi dalam volatilitas dan suku bunga bebas risiko yang bervariasi.

Kata Kunci: *Persamaan Black-Scholes, Produk opsi, BSDE, Teorema Feynman-Kac, Analisis sensitivitas*

Introduction

Investment refers to the allocation of capital in a company or project with the expectation of generating profit. The rapid development of financial markets has expanded investment alternatives, enabling investors to either purchase shares directly in the capital market or acquire derivative assets[1]. Derivatives are financial instruments whose value depends on an underlying asset and are commonly employed by market participants to hedge portfolio risk. Compared to their underlying assets, derivatives are generally less costly and thus widely utilized[2]. One of the most well-known derivative products is options. The option is defined as a contract granting the right to buy or sell an underlying asset under specified conditions. Options are further classified into call and put options according to the rights they confer, and into European or American types based on their exercise rules [3].

The pricing of options is most often derived from the Black–Scholes equation [4]. While the Black–Scholes model has been studied extensively through classical approaches based on stochastic calculus and partial differential equations (PDEs), alternative formulations using Backward Stochastic Differential Equations (BSDEs) have received increasing attention in recent decades. El Karoui et al. established a foundational framework for applying BSDEs in finance, demonstrating their flexibility for option pricing and hedging[5]. Cordoni and Di Persio further developed this perspective by applying BSDEs to hedging strategies, option pricing, and insurance problems[6], while Huijskens introduced numerical methods for solving BSDEs with financial applications[7]. Together, these studies underline the versatility of BSDEs in financial modeling.

Despite these advances, the derivation of the Black–Scholes equation through BSDEs remains relatively uncommon. Compared to the traditional PDE-based approach, BSDEs offer several advantages, such as providing a natural framework for addressing incomplete markets where not all options are replicable, and avoid the technical requirement of transforming the probability measure into the risk-neutral framework[6], [7]. These features suggest that BSDEs can simplify and generalize option pricing, offering both theoretical and practical benefits.

This study derives the Black–Scholes equation through BSDEs by modeling the wealth dynamics of an investor and applying the analogy of the Feynman–Kac theorem[7], [8]. After the Black-Scholes model has obtained, then solved analytically via variable transformations to construct a diffusion equation with payoff-based initial conditions. A variety of numerical solution methods for the Black–Scholes equation have become well established in the literature, such as the binomial tree method[4], [9], finite difference schemes[4], [9], [10], and finite element/fractional discretization approaches [11]. The solution for call options is presented in detail, while put option pricing is obtained through the put–call parity principle. Finally, the analytical solutions are implemented in MATLAB to simulate option price behavior, with particular emphasis on the effects of volatility and the risk-free interest rate.

The structure of this paper is as follows. Section 2 introduces the formulation of Backward Stochastic Differential Equations. Section 3 discusses the connection between BSDEs and the Feynman–Kac theorem. Section 4 presents the derivation and analytical

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solution of the Black–Scholes equation. Section 5 provides numerical simulations and analysis of option price behavior under varying parameters and finally conclusion in Section 6.

The BSDEs for Option Pricing

Considering a financial market with risk asset $S(t)$ and risk-free asset $B(t)$ that satisfies the following differential equations,

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t), \quad (1)$$
$$dB(t) = rB(t)dt. \quad (2)$$

An investor's wealth Y is assumed to consist of a single risky asset and a single risk-free asset, where the risky asset corresponds to a stock and the risk-free asset is represented by a bank savings account. Let Δ_s denote the number of shares of the risky asset S, and Δ_b denote the amount invested in the risk-free asset B. Then, the investor's wealth at time t can be expressed as[4]

$$Y(t) = \Delta_s(t)\, S(t) + \Delta_b(t)B(t)$$

The investor's wealth is assumed to follow a self-financing strategy, meaning that changes in the value of the portfolio are solely due to the gains or losses generated by the assets held within the portfolio. Consequently, the dynamics of the investor's wealth, often referred to as the wealth process, can be written in differential form as

$$dY(t) = \Delta_s\, dS(t) + \Delta_b dB(t) + S(t)d\Delta_s + B(t)d\Delta_b.$$

However, the holdings are constant on the interval $(t, t + dt)$, where it means that the investor holds do not change on that interval. Hence the processes Δ_s and Δ_b are locally constant, so their differentials vanish:

$$\Delta_s = 0; \quad \Delta_b = 0; \text{ on } (t, t + dt).$$

Consequently, the wealth dynamics reduce to

$$dY(t) = \Delta_s\, dS(t) + \Delta_b dB(t). \quad (3)$$

Substituting Equation (1) and Equation (2) into Equation (3), we obtain,

$$dY(t) = (rY(t) + \theta Z)dt + Z\, dW(t). \quad (4)$$

Where $Y(0) = y_0$, $Z = \Delta_s \sigma\, S$ and $\theta = (\mu - r)/\sigma$. Now, let us define that $f(t, Y(t), Z) = -(rY(t) + \theta Z)$, later the function f is called by a generator of the BSDE and considering the terminal condition of the call option price, $\xi = \max(S(T) - K, 0)$ where K denotes a strike price. To determine the call option price, we have to find a process of Z, and the initial value y_0 that satisfies Equation (4) so that $Y(T) = \xi$. Therefore, the call option price is a pair of solution $(Y(t), Z)$ obtained by solving the following BSDE:

$$dY(t) = -f(t, Y(t), Z)dt + Z\, dW(t), \quad (5)$$
$$Y(T) = \xi. \quad (6)$$

Linking The BSDEs to Semi-linear PDEs

Let us consider a semi-linear PDEs in n-dimensional case, $x \in \mathbb{R}^n$,

$$-\frac{\partial v(x,t)}{\partial t} - \mathcal{L}v - f(t, x, v, \sigma^T D_x v) = 0, \quad (7)$$

$$v(x, T) = \xi(x)$$

where $\mathcal{L}$ is an infinitesimal generator function given by

$$\mathcal{L}v = \mu(x,t)D_x v + \frac{1}{2} Tr(\sigma(x,t)\sigma^T(x,t)D_x^2 v),$$

$\sigma^T(x, t)$ and $D_x^2 v$ denote $\frac{\partial v}{\partial x}$ and $\frac{\partial^2 v}{\partial x^2}$ respectively. In one dimensional case, it can be reduced as

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$$-\frac{\partial v}{\partial t}-\mu(x,t)\frac{\partial v}{\partial x}-\frac{1}{2}\sigma^2(x,t)\frac{\partial^2 v}{\partial x^2} - f\left(t,x,v,\sigma(x,t)\ \frac{\partial v}{\partial x}\right)=0. \quad (8)$$

The relation between the BSDEs and the semi-linear PDEs can be explained by using the Feynman-Kac theorem[7], [8].

Theorem 1

Let $v(x,t)$ is a solution to Equation (7) and let

$$Y(t)=v(x,t);\quad Z(t)=\sigma^T(x,t)D_x v.$$

Then, (Y,Z) is a pair of solution to the BSDE

$$dY(t)=-f\big(t,x(t),Y(t),Z(t)\big)dt+Z(t)dW(t)$$

with the generator function, $f\big(t,x(t),Y(t),Z(t)\big)$, the terminal conditions $\xi=v(x,T)$ and the forward SDEs

$$dY(t)=\mu(x,t)dt+\sigma(x,t)dW(t).$$

▪

Now, let us consider the BSDE in Equation (5) and Equation (6). We found that $f(t,Y(t),Z(t))=-rY(t)-\theta Z(t)$ and $Z(t)=\sigma\, S(t)\Delta_s$. Under a replicating strategy framework, where the portfolio is constructed to ensure that the wealth process replicates the option price at any time, Δ_s is defined as a ratio between the changed of option price over the underlying asset, $\Delta_s=\frac{\partial v}{\partial S}$ [4]. As a result, the replicating portfolio eliminates arbitrage opportunities and minimizes risk.

Based on Theorem 1, the pair

$$Y(t)=v(S,t);\quad Z(t)=\sigma S\frac{\partial v}{\partial S},$$

is a solution to the system of Equation (1) and Equation (5) with the terminal condition $v(S,T)=\max(S-K,0)$, where the function $v(S,t)$ is a solution to the following semi-linear PDE

$$-\frac{\partial v}{\partial t}-\mu S\frac{\partial v}{\partial S}-\frac{1}{2}\sigma^2 S^2\frac{\partial^2 v}{\partial S^2} - f\left(t,S,v,\sigma\ \frac{\partial v}{\partial S}\right)=0 \quad (9)$$

with

$$f\left(t,S,v,\sigma\ \frac{\partial v}{\partial S}\right)=-rv(x,t)-S(\mu-r)\frac{\partial v}{\partial S}.$$

After simplification on Equation (9), we obtain the well-known equation, the Black-Scholes equation as follows:

$$-\frac{\partial v}{\partial t}-rS\frac{\partial v}{\partial S}-\frac{1}{2}\sigma^2 S^2\frac{\partial^2 v}{\partial S^2}+rv=0 \quad (10)$$

with the terminal condition,

$$v(S,T)=\max(S-K,0).$$

The Formula for the European Option Pricing

In the previous section, we have denoted the option price by $v(S,t)$ in Equation (10). However, in case of the call option price, we decide to denote the call option price by $C(S,t)$. So, Equation (10) is rewritten by

$$\frac{\partial C}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}+\ rS\frac{\partial C}{\partial S}-rC=0, \quad (11)$$

with the terminal conditions is given by

$$C(S,T)=\begin{cases}S-K, & \text{for } S>K\\ 0, & \text{for } S\le K\end{cases}. \quad (12)$$

To solve Equation (11) with the terminal condition (12), firstly we need to transform these equations into the dimensionless form by following this variables,

$$C(S,t)=Kv(x,\tau);\quad S=Ke^x;\ \ t=T-\frac{2\,\tau}{\sigma^2}.$$

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Using the total derivation, we obtain

$$-\frac{\partial v}{\partial \tau}+\frac{\partial^2 v}{\partial x^2}+\left(\frac{2r}{\sigma^2}-1\right)\frac{\partial v}{\partial x}-\frac{2r}{\sigma^2}v=0,$$

and the terminal condition is converted to the initial condition as follows:

$$v(x,0)=\begin{cases} e^x-1, & \text{for } x>0 \\ 0, & \text{for } x\le 0 \end{cases}.$$

After that, we introduce the following transformation,

$$v(x,\tau)=e^{\alpha x+\beta\tau}u(x,\tau),$$

with $\alpha=-\frac{1}{2}(k-1)$ and $\beta=-\frac{1}{4}(k+1)^2$, and $k=2r/\sigma^2$. So, we obtain the diffusion equation as follows

$$\frac{\partial u}{\partial \tau}-\frac{\partial^2 u}{\partial x^2}=0, \tag{13}$$

with the initial condition,

$$u(x,0)=\begin{cases} e^{\frac{1}{2}(k+1)x}-e^{\frac{1}{2}(k-1)x}, & \text{for } x>0 \\ 0, & \text{for } x\le 0. \end{cases} \tag{14}$$

Considering the fundamental of diffusion solution[12], [13], we can write the solution of Equation (13) with the initial condition, (14) is as follows

$$u(x,\tau)=\frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty}u(s,0)e^{-\frac{(x-s)^2}{4\tau}}ds.$$

Assuming that

$$q=\frac{x-s}{\sqrt{2\tau}};\qquad s=x+q\sqrt{2\tau};\qquad ds=\sqrt{2\tau}\,dq,$$

then $u(x,\tau)$ becomes

$$u(x,\tau)=\frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty}u\left(x+q\sqrt{2\tau},0\right)e^{-\frac{1}{2}q^2}\sqrt{2\tau}\,dq$$

$$=\frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{-\frac{x}{\sqrt{2\tau}}}u\left(x+q\sqrt{2\tau},0\right)e^{-\frac{1}{2}q^2}dq\right)+\frac{1}{\sqrt{2\pi}}\left(\int_{-\frac{x}{\sqrt{2\tau}}}^{\infty}u\left(x+q\sqrt{2\tau},0\right)e^{-\frac{1}{2}q^2}dq\right) \tag{15}$$

and substitute the initial value (14) into Equation (15), we obtain

$$u(x,\tau)=\frac{1}{\sqrt{2\pi}}\left(0+\int_{-\frac{x}{\sqrt{2\tau}}}^{\infty}u\left(x+q\sqrt{2\tau},0\right)e^{-\frac{1}{2}q^2}dq\right)$$

$$=\frac{1}{\sqrt{2\pi}}\left(\int_{-\frac{x}{\sqrt{2\tau}}}^{\infty}e^{\frac{1}{2}(k+1)\left(x+q\sqrt{2\tau}\right)}e^{-\frac{1}{2}q^2}dq\right)$$

$$-\frac{1}{\sqrt{2\pi}}\left(\int_{-\frac{x}{\sqrt{2\tau}}}^{\infty}e^{\frac{1}{2}(k-1)\left(x+q\sqrt{2\tau}\right)}e^{-\frac{1}{2}q^2}dq\right).$$

For convenience, we assume that

$$I_1=\int_{-\frac{x}{\sqrt{2\tau}}}^{\infty}e^{\frac{1}{2}(k+1)\left(x+q\sqrt{2\tau}\right)}e^{-\frac{1}{2}q^2}dq\,;$$

$$I_2=\int_{-\frac{x}{\sqrt{2\tau}}}^{\infty}e^{\frac{1}{2}(k-1)\left(x+q\sqrt{2\tau}\right)}e^{-\frac{1}{2}q^2}dq,$$

and solving the integral partially.

Now, solving I_2 with the following steps:

$$I_2=\frac{1}{\sqrt{2\pi}}\left(\int_{-\frac{x}{\sqrt{2\tau}}}^{\infty}e^{\frac{1}{2}(k-1)\left(x+q\sqrt{2\tau}\right)}e^{-\frac{1}{2}q^2}dq\right)$$

$$=\frac{e^{\frac{1}{2}(k-1)x}}{\sqrt{2\pi}}\left(\int_{-\frac{x}{\sqrt{2\tau}}}^{\infty}e^{-\frac{1}{2}\left(q^2-(k-1)q\sqrt{2\tau}\right)}e^{\frac{1}{4}(k-1)^2\tau-\frac{1}{4}(k-1)^2\tau}dq\right)$$

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$$= \frac{e^{\frac{1}{2}(k-1)x+\frac{1}{4}(k-1)^2\tau}}{\sqrt{2\pi}}\left(\int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{-\frac{1}{2}\left(q-\frac{1}{2}(k-1)\sqrt{2\tau}\right)^2} dq\right)$$

$$= e^{\frac{1}{2}(k-1)x+\frac{1}{4}(k-1)^2\tau} N\left(\frac{x-(k-1)}{\sqrt{2\tau}}\right),$$

where $N(.)$ is a cumulative distribution function of standard normal distributions. Using same procedure to I_2, we can obtain the integral solution of I_1. So,

$$u(x,\tau) = e^{\frac{1}{2}(k+1)x+\frac{1}{4}(k+1)^2\tau} N\left(\frac{x-(k+1)}{\sqrt{2\tau}}\right) - e^{\frac{1}{2}(k-1)x+\frac{1}{4}(k-1)^2\tau} N\left(\frac{x-(k-1)}{\sqrt{2\tau}}\right)$$

Finally, by transforming into the original variables, we obtain the formula for call option price as follows:

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2), \qquad (16)$$

where,

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{(T-t)}},$$

$$d_2 = \frac{\ln\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{(T-t)}}.$$

While the formula for the call option price was obtained as Equation (16), the formula for the put option price can be derived directly from the put-call parity principle. The Put–call parity is a fundamental no-arbitrage relationship between the European call and put option written on the same asset with identical strike price and maturity[3], [4]. It can be stated that the difference between call and put option price equals the current asset price minus the present value of the strike price,

$$C(S,t) - P(S,t) = S - Ke^{-r(T-t)}.$$

As a result, the put option price can be formulated as

$$P(S,t) = Ke^{-\tau(T-t)}N(-d_2) - SN(-d_1). \qquad (17)$$

Simulation

We have obtained the formula for the call and put option price as Equation (16) and Equation (17), respectively. This section presents the simulation of the option price given by some parameters. Before conducting the simulation, we set up the following parameters: the strike price $(K) = 100$, volatility $(\sigma) = 0.1$, 0.3, and 0.5, risk free interest rate (r) = 1%, 3%, and 5%, and the maturity time is 1 year.

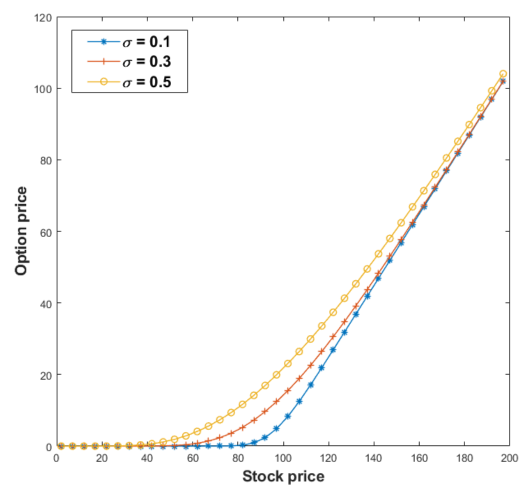

Figure 1. The effect of volatility to the call option price

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The first simulation is conducted to examine the effect of stock return volatility on the prices of call and put options. The relationship between option prices and volatility is illustrated in Figures 1 and 2. As shown in Figure 1, the call option value reflects its payoff structure. When the stock price S is below the strike price K, the option is out of the money and its value remains close to zero, but once S exceeds K, the call price rises. The impact of volatility is evident, which means that higher volatility shifts the increase in option value to lower stock prices, since the probability of ending in the money is greater.

For the put option in Figure 2, the value increases as the stock price falls below the strike price.

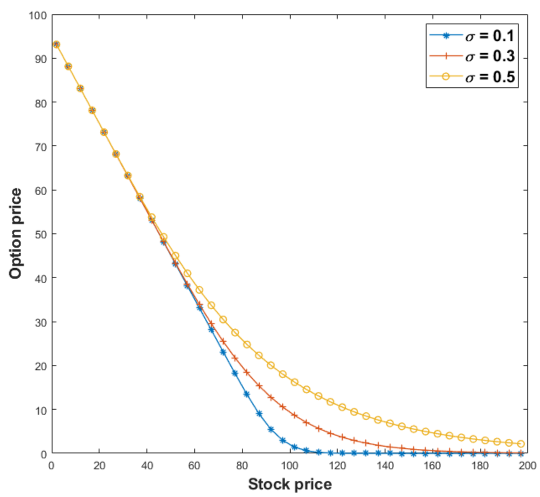

Figure 2. The effect of volatility to the put option price

Under low volatility, the put price rises more predictably as S decreases, making immediate exercise attractive once the stock falls below K. In contrast, under higher volatility, investors may delay exercise since upward price movements could still generate profit opportunities, and the higher uncertainty also raises the option's time value. Overall, greater volatility increases both call and put option values, reflecting the higher likelihood of significant stock price movements before maturity.

The next simulation investigates how changes in the risk-free interest rate affect option values. The results, displayed in Figures 3 and 4. It shows an opposite result between call and put options.

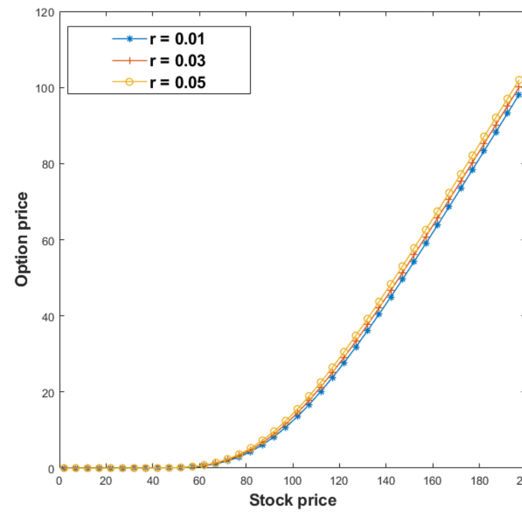

Figure 3. The effect of risk-free interest rate to the call option price

For call options in Figure 3, higher the interest rate increases the call option price. In other words, the effect arises from the time value of money, as a higher risk-free rate increases the present value growth of the underlying asset. As a result, call option prices increase with higher interest rates. In contrast, higher the interest rate decreases the put option price shown in Figure 4.

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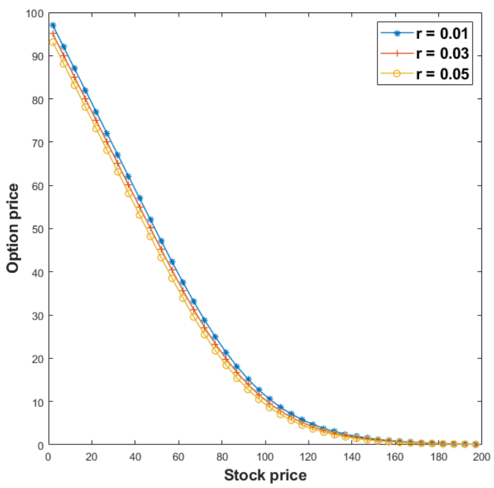

Figure 4. The effect of risk-free interest rate to the put option price

A higher risk-free interest rate makes owning the underlying asset more interesting. As a result, increasing the interest rate not only reduces the need for a put option protection but also decreases the present value of the future sale at the strike price.

Conclusion

Based on the analysis and discussion, several conclusions can be drawn. First, the Black–Scholes equation for option pricing can be derived more directly and intuitively by applying the Feynman–Kac theorem, which establishes the connection between backward stochastic differential equations (BSDEs) and partial differential equations (PDEs). The option price itself can be obtained analytically by solving the Black–Scholes. Furthermore, the relationship between call and put options is characterized by the principle of put–call parity. In terms of parameter effects, higher volatility increases the value of call options as well as put options. Conversely, a lower risk-free interest rate tends to increase the value of put options, whereas it reduces the value of call options.

Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this work.

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